CURVILINEAR SCHEMES AND MAXIMUM RANK OF FORMS

EDOARDO BALLICO AND ALESSANDRA BERNARDI

Abstract. We define the *curvilinear rank* of a degree d form P in $n+1$ variables as the minimum length of a curvilinear scheme, contained in the d-th Veronese embedding of $\mathbb{P}^n$, whose span contains the projective class of P. Then, we give a bound for rank of any homogenous polynomial, in dependance on its curvilinear rank.

Introduction

The *rank* $\mathrm{r}(P)$ of a homogeneous polynomial $P \in \mathbb{C}[x_0, \dots, x_n]$ of degree d, is the minimum $r \in \mathbb{N}$ such that P can be written as sum of r pure powers of linear forms $L_1, \dots, L_r \in \mathbb{C}[x_0, \dots, x_n]$:

$$P = L_1^d + \cdots + L_r^d. \tag{1}$$

A very interesting open question is to determine the maximum possible value that the rank of a form (i.e. a homogeneous polynomial) of given degree in a certain number of variables can have. On our knowledge, the best general achievement on this problem is due to J.M. Landsberg and Z. Teitler that in [14, Proposition 5.1] proved that the rank of a degree d form in $n+1$ variables is smaller or equal than $\binom{n+d}{d} - n$. Unfortunately this bound is sharp only for $n = 1$ if $d \geq 2$; in fact, for example, if $n = 2$ and $d = 3, 4$, then the maximum ranks are 5 and 7 respectively (see [6, Theorem 40 and 44]).

Few more results were obtained by focusing the attention on limits of forms of given rank. When a form P is in the Zariski closure of the set of forms of rank s, it is said that P has *border rank* $\underline{\mathrm{r}}(P)$ equal to s. For example, the maximum rank of forms of border ranks 2, 3 and 4 are known (see [6, Theorems 32 and 37] and [2, Theorem 1]). In this context, in [1] we posed the following:

Question 1 ([1])**.** Is it true that $\mathrm{r}(P) \leq d(\underline{\mathrm{r}}(P) - 1)$ for all degree d forms P? Moreover, does the equality hold if and only if the projective class of P belongs to the tangential variety of a Veronese variety?

The Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}}$, with $n \geq 1$, $d \geq 2$ and $N_{n,d} := \binom{n+d}{d} - 1$ is the classical d-uple Veronese embedding $\nu_d : \mathbb{P}^n \to \mathbb{P}^{N_{n,d}}$ and parameterizes projective classes of degree d pure powers of linear forms in $n+1$ variables. Therefore the rank $\mathrm{r}(P)$ of $[P] \in \mathbb{P}^{N_{n,d}}$ is the minimum r for which there exists a smooth zero-dimensional scheme $Z \subset X_{n,d}$ whose span contains $[P]$

1991 *Mathematics Subject Classification.* 14N05.

Key words and phrases. Maximum rank, curvilinear rank, curvilinear schemes, cactus rank.

The authors were partially supported by CIRM of FBK Trento (Italy), Mathematical Department of Trento (Italy), Project Galaad of INRIA Sophia Antipolis Méditerranée (France), Marie Curie: Promoting science (FP7-PEOPLE-2009-IEF), MIUR and GNSAGA of INdAM (Italy).

(with an abuse of notation we are extending the definition of rank of a form P given in (1) to its projective class $[P]$). More recently, other notions of polynomial rank have been introduced and widely discussed ([8], [15], [7], [5], [3]). They are all related to the minimal length of a certain zero-dimensional schemes embedded in $X_{m,d}$ whose span contains the given form. Here we recall only the notion of *cactus rank* $\mathrm{cr}(P)$ of a form P with $[P] \in \mathbb{P}^{N_{n,d}}$ (in [15], [7], [5] and also in [12, Definition 5.1] as "scheme length"):

$$\mathrm{cr}(P) = \min\{\deg(Z) \mid Z \subset X_{n,d},\ \dim_K Z = 0 \text{ and } [P] \in \langle Z \rangle\}.$$

With this definition, it seems more reasonable to state Question 1 as follows:

Question 2. Fix $[P] \in \mathbb{P}^{N_{m,d}}$ with $\mathrm{r}(P) > 0$. Is it true that $\mathrm{r}(P) \le (\mathrm{cr}(P) - 1)d$?

In this paper we want to deal with a more restrictive bur more wieldy notion of rank, namely the "curvilinear rank". We say that a scheme $Z \subset \mathbb{P}^N$ is *curvilinear* if it is a finite union of schemes of the form $\mathcal{O}_{C_i,P_i}/\mathfrak{m}_{P_i}^{e_i}$ for smooth points P_i on reduced curves $C_i \subset \mathbb{P}^N$, or equivalently that the tangent space at each connected component of Z supported at the P_i's has Zariski dimension ≤ 1. We define the *curvilinear rank* $\mathrm{Cr}(P)$ of a degree d form P in $n+1$ variables as:

$$\mathrm{Cr}(P) := \min\{\deg(Z) \mid Z \subset X_{n,d},\ Z \text{ curvilinear},\ [P] \in \langle Z \rangle\}.$$

The main result of this paper is the following:

Theorem 1. *For any degree d form P we have that*

$$\mathrm{r}(P) \le (\mathrm{Cr}(P) - 1)d + 2 - \mathrm{Cr}(P).$$

Theorem 1 is sharp if $\mathrm{Cr}(P) = 2, 3$ ([6, Theorem 32 and 37]).

The next question will be to understand if Theorem 1 holds even though we substitute the curvilinear rank with the cactus rank:

Question 3. Fix $[P] \in \mathbb{P}^{N_{m,d}}$ with $\mathrm{r}(P) > 0$. Is it true that $\mathrm{r}(P) \le (\mathrm{cr}(P) - 1)d + 2 - \mathrm{cr}(P)$?

This manuscript is organized as follows: Section 1 is entirely devoted to the proof of Theorem 1 with two auxiliary lemmas; in Section 2 we study the case of ternary forms and we prove that, in such a case, Question 2 has an affirmative answer.

1. Proof of Theorem 1

Let us begin this section with some Lemmas that will allow us to give a lean prof of the main theorem.

We say that an irreducible curve T is *rational* if its normalization is a smooth rational curve.

Lemma 1. *Let $Z \subset \mathbb{P}^N$ be a zero-dimensional curvilinear scheme of degree k. Then there is an irreducible and rational curve $T \subset \mathbb{P}^N$ such that $\deg(T) \le k - 1$ and $Z \subset T \subseteq \langle Z \rangle$.*

Proof. If the scheme Z is in linearly general position, namely $\langle Z\rangle \simeq \mathbb{P}^{k-1}$, then there always exists a rational normal curve of degree $k-1$ passing through it (this is a classical fact, see for instance [11, Theorem 1]). If Z is not in linearly general position, consider $\mathbb{P}(H^0(Z,\mathcal{O}_Z(1))) \simeq \mathbb{P}^{k-1}$. In such a $\mathbb{P}^{k-1}$ there exists a curvilinear scheme W of degree k in linearly general position such that the projection $\ell_V : \mathbb{P}^{k-1} \setminus V \to \langle Z\rangle$ from a $(k-\dim(\langle Z\rangle)-1)$-dimensional vector space V induces an isomorphism between W and Z. Consider now the degree $k-1$ rational normal curve $C \subset \mathbb{P}^{k-1}$ passing through W, its projection $\ell_V(C)$ contains Z and it is irreducible and rational since C is irreducible and rational and, by construction, $\deg(\ell_V(C)) \le \deg(C) = k-1$. □

In the following lemma we will use the notion of X*-rank* of a point $P \in \langle X\rangle$ with respect to a variety X; we indicate it with $\mathrm{r}_X(P)$ and it represents the minimum number of points $P_1,\dots,P_s \in X$ whose span contains P and we will say that the set $\{P_1,\dots,P_s\}$ *evinces* P.

Lemma 2. *Let $Y \subset \mathbb{P}^N$ be an integral and rational curve of degree d. Fix $P \in \langle Y\rangle$ and assume the existence of a curvilinear degree k scheme $Z \subset Y$, with $d \ge k \ge 2$, such that $P \in \langle Z\rangle$ and $P \notin \langle Z'\rangle$ for any $Z' \subsetneq Z$. If $k \le (d+2)/2$, then $\mathrm{r}_Y(P) \le d+2-k$, otherwise $\mathrm{r}_Y(P) \le k$.*

Proof. If Y is a rational normal curve, then this is weak version of a celebrated theorem of Sylvester (cfr. [10], [14, Theorem 5.1], [6, Theorem 23]). Hence we may assume $d > \dim\langle Y\rangle$. Observe that the hypothesis $P \in \langle Z\rangle$ and $P \notin \langle Z'\rangle$ for any $Z' \subsetneq Z$, allows to say that the dimension of $\langle Z\rangle$ is $k-1$, i.e. Z is linearly independent, therefore $\dim\langle Z'\rangle = \deg(Z')-1$ for every $Z' \subset Z$. This allows us to consider a $(d-\dim\langle Y\rangle-1)$-dimensional linear subspace $V \subset \mathbb{P}^d$ and a rational normal curve $C \subset \mathbb{P}^d$ of degree d such that $V \cap C = \emptyset$ and the linear projection $\ell_V : \mathbb{P}^d \to \langle Y\rangle$ from a V is surjective. Moreover it also assures the existence of a scheme $U \subset C$ such that $\ell_V(U) = Z$ is a degree k effective divisor of C that spans a $\mathbb{P}^{k-1}$ which doesn't intersect V. Hence ℓ_V induces an isomorphism $\phi : \langle U\rangle \to \langle Z\rangle$. Let $O \in \langle U\rangle$ be the only point such that $\phi(O) = P$. Let $S_1 \subset C$ be the set of points evincing $\mathrm{r}_C(O)$ and set $S := \ell_V(S_1) \subset Y$. Now, the crucial observations are that $\sharp(S) \le \sharp(S_1)$ and $P \in \langle S_1\rangle$. Therefore $\mathrm{r}_Y(P) \le \mathrm{r}_C(O)$. Now, by [6, Theorem 23], we have that if $k \le (d+2)/2$ then $\mathrm{r}_C(O) = d+2-k$, if $k > (d+2)/2$ then either $\mathrm{r}_C(O) = d+2-k$ or $\mathrm{r}_C(O) = k$. □

We are now ready to prove the main theorem of this paper.

Proof of Theorem 1: Let $Z \subset X_{n,d}$ be a minimal degree curvilinear scheme such that $P \in \langle Z\rangle$, and let $U \subset \mathbb{P}^n$ be the curvilinear scheme such that $\nu_d(U) = Z$. Say that of degree $\mathrm{Cr}(P) = \deg(Z) = \deg(U) := k \ge 2$

By Lemma 1, there exists a rational curve $T \subset \mathbb{P}^n$ such that $U \subset T$ and $\deg(T) \le k-1$. The curve $\nu_d(T)$ is an irreducible rational curve of degree $d \cdot \deg(T) \le d(k-1)$, and obviously $P \in \langle \nu_d(T)\rangle$, hence the integer $\mathrm{r}_{\nu_d(T)}(P)$ is well-defined. Now, since $\nu_d(T)$ is an integral curve of degree $\le d(k-1)$ it spans a projective space of dimension $\le d(k-1)$ (this is a weak form of Riemann-Roch), therefore P, which belongs to this span, has

$$\mathrm{r}_{\nu_d(T)}(P) \le \dim\langle \nu_d(T)\rangle \le d(k-1) \qquad (2)$$

([14, Proposition 4.1] or [9, Lemma 8.2]).
Since $k \geq 2$, the function $t \mapsto d(t-1)+2-t$ is increasing for $t > 0$ and every subscheme of a curvilinear scheme is curvilinear, we may assume $P \notin \langle Z' \rangle$ for any $Z' \subsetneq Z$.
To conclude our prove it is sufficient to apply Lemma 2 to the integral rational curve $\nu_d(T)$ and get

$$\mathrm{r}_{\nu_d(T)}(P) \leq d(k-1)+2-k.$$

Now the rank $\mathrm{r}(P)$ that we want to estimate is nothing else than $\mathrm{r}_{X_{n,d}}(P)$, and, since $\nu_d(T) \subset X_{n,d}$, we obviously have that $\mathrm{r}(P) \leq \mathrm{r}_{\nu_d(T)}(P)$. □

2. Superficial case

In this section we show that Question 2 has an affirmative answer in the case $m = 2$ of ternary forms. More precisely, we prove the following result.

Proposition 1. *Let P be a ternary form of degree d with $\mathrm{Cr}(P) \geq 2$. Then $\mathrm{r}(P) \leq (\mathrm{Cr}(P)-1)d$.*

Before giving the proof of Proposition 1, we need the following result.

Proposition 2. *Let $Z \subset \mathbb{P}^2$ be a degree $k \geq 4$ zero-dimensional scheme. There is an integral curve $C \subset \mathbb{P}^2$ such that $\deg(C) = k-1$ and $Z \subset C$ if and only if Z is not contained in a line.*

Proof. First of all, if Z is contained in a line D, we may even find a smooth curve $C \subset \mathbb{P}^2$ such that $C \cap D = Z$ as schemes (this is easy to check by using the homogeneous equations of D and C). We assume therefore that D is not contained in a line.

Claim 1. The linear system $|\mathcal{I}_Z(k-1)|$ has no base points outside Z_{red}.

Proof of Claim 1. Fix $P \in \mathbb{P}^2 \backslash Z_{red}$. Since $\deg(Z \cup \{P\}) = k+1$, we have $h^1(\mathcal{I}_{Z\cup\{P\}}(k-1)) > 0$ if and only if there is a line D containing $Z \cup \{P\}$, but, since in our case Z is not contained in line, we get $h^1(\mathcal{I}_{Z\cup\{P\}}(k-1)) = 0$. Hence $h^0(\mathcal{I}_{Z\cup\{P\}}(k-1)) = h^0(\mathcal{I}_Z(k-1)) - 1$, i.e. P is not a base point of $|\mathcal{I}_Z(k-1)|$.

By Claim 1, the linear system $|\mathcal{I}_Z(k-1)|$ induces a morphism $\psi : \mathbb{P}^2 \setminus Z_{red} \to \mathbb{P}^x$.

Claim 2. We have $\dim(\psi) = 2$.

Proof of Claim 2. It is sufficient to prove that the differential $d\psi(Q)$ of ψ has rank 2 for a general $Q \in \mathbb{P}^2$. Assume that $d\psi(Q)$ has rank ≤ 1, i.e. assume the existence of a tangent vector $\mathbf{v}$ at Q in the kernel of the linear map $d\psi(Q)$. Since $h^1(\mathcal{I}_{Z\cup\{P\}}(k-1)) = 0$ (see proof of Claim 1), this is equivalent to $h^1(\mathcal{I}_{Z\cup\mathbf{v}}(k-1)) > 0$. Since $\deg(Z \cup \mathbf{v}) = k+2 \leq 2(k-1)+1$, there is a line $D \subset \mathbb{P}^2$ such that $\deg(D \cap (Z \cap \mathbf{v}) \geq k+1$. Hence $\deg(Z \cap D) \geq k-1$. Since $k \geq 4$ there are at most finitely many lines $D_1, \dots, D_s$ such that $\deg(D_i \cap Z) \geq k-1$ for all i. If $Q \notin D_1 \cup \cdots \cup D_s$, then $\deg(D \cap (Z \cap \mathbf{v}) \leq k$ for every line D.

By Claim 2 and Bertini's second theorem ([13, Part 4 of Theorem 6.3]) a general $C \in |\mathcal{I}_Z(k-1)|$ is irreducible. □

Any degree 2 zero-dimensional scheme $Z \subset \mathbb{P}^n$, $n \geq 2$ is contained in a unique line and hence it is contained in a unique irreducible curve of degree $2-1$. Now we check that in case our form has curvilinear rank equal to 3, then Proposition 2 fails in a unique case.

Remark 1. Let $Z \subset \mathbb{P}^2$ be a zero-dimensional scheme such that $\deg(Z) = 3$. Since $h^1(\mathcal{I}_Z(2)) = 0$ ([6], Lemma 34), we have $h^0(\mathcal{I}_Z(2)) = 3$. A dimensional count gives that Z is not contained in a smooth conic if and only if there is $P \in \mathbb{P}^2$ with $Z = 2P$ (in this case $|\mathcal{I}_Z(2)|$ is formed by the unions $R \cup L$ with R and L lines through P).

We conclude our paper with the Proof of Proposition 1.

Proof of Proposition 1. If : $\mathrm{Cr}(P) = 2, 3$, then the statement is true by [6, Theorems 32 and 37]. If $\mathrm{Cr}(P) \geq 4$, then we can repeat the proof of Theorem 1 until (2) by using as curve T appearing in Theorem 1, the curve C of Proposition 2. $\square$

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Dept. of Mathematics, University of Trento, 38123 Povo (TN), Italy
E-mail address: edoardo.ballico@unitn.it

GALAAD, INRIA Méditerranée, BP 93, 06902 Sophia Antipolis, France.
E-mail address: alessandra.bernardi@inria.fr